Unit - V

Short answer questions.

1. If $\omega = (y-z)(z-x)(x-y)$ find the value of $\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} + \frac{\partial \omega}{\partial z}$

Sol. Given $\omega = (y-z)(z-x)(x-y)$

$$\omega = \cancel{xyz} - x^2y - xz^2 + x^2z - y^2z + y^2x + yz^2 - \cancel{xyz}$$

$$\omega = x^2z + y^2x + yz^2 - x^2y - xz^2 - y^2z$$

$$\frac{\partial \omega}{\partial x} = 2zx + y^2(1) + 0 - 2xy - z^2 - 0$$

$$\frac{\partial \omega}{\partial x} = y^2 + 2xz - 2xy - z^2$$

$$\frac{\partial \omega}{\partial y} = 0 + 2yx + z^2 - x^2(1) - 0 - 2yz$$

$$\frac{\partial \omega}{\partial y} = 2xy + z^2 - x^2 - 2yz$$

$$\frac{\partial \omega}{\partial z} = x^2(1) + 0 + y(2z) - 0 - x(2z) - y^2(1)$$

$$\frac{\partial \omega}{\partial z} = x^2 + 2yz - 2xz - y^2$$

$\therefore$ consider $\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} + \frac{\partial \omega}{\partial z}$

$$= \cancel{y^2} + \cancel{2xz} - \cancel{2xy} - \cancel{z^2} + \cancel{2xy} + \cancel{z^2} - \cancel{x^2} - \cancel{2yz} + \cancel{x^2} + \cancel{2yz} - \cancel{2xz} - \cancel{y^2}$$

$$= 0 \quad //$$

2. verify Eulers theorem for $z = ax^2 + 2hxy + by^2$.

Sol. Given $z = ax^2 + 2hxy + by^2$

$$f(x, y) = ax^2 + 2hxy + by^2$$

$$f(kx, ky) = a(kx)^2 + 2h(kx)(ky) + b(ky)^2$$

$$= ak^2x^2 + 2hk^2xy + bk^2y^2$$

$$= k^2 [ax^2 + 2hxy + by^2]$$

$$= k^2 (z)$$

$$f(kx, ky) = k^2 [f(x,y)]$$

$\therefore$ Given function is a homogenious function of degree 2.

Now we have to prove that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$

$$z = ax^2 + 2hxy + by^2$$

$$\frac{\partial z}{\partial x} = a(2x) + 2h(1)y + b(0)$$

$$= 2ax + 2hy$$

$$\frac{\partial z}{\partial y} = 0 + 2hx(1) + b(2y)$$

$$= 2by + 2hx.$$

$\therefore$ Consider LHS $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$

$$= x(2ax + 2hy) + y(2by + 2hx)$$

$$= 2ax^2 + 2hxy + 2by^2 + 2hxy$$

$$= 2ax^2 + 4hxy + 2by^2$$

$$= 2(ax^2 + 2hxy + by^2)$$

$$= 2z \;//$$

Hence Eulers Theorem is verified.

3. If $u = x^2 - 2y$ $v = x+y+z$ $w = x - 2y + 3z$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Sol.

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$u = x^2 - 2y$ $\qquad v = x+y+z$ $\qquad w = x - 2y + 3z$

$\frac{\partial u}{\partial x} = 2x$ $\qquad \frac{\partial v}{\partial x} = 1$ $\qquad \frac{\partial w}{\partial x} = 1$

$\frac{\partial u}{\partial y} = -2$ $\qquad \frac{\partial v}{\partial y} = 1$ $\qquad \frac{\partial w}{\partial y} = -2$

$\frac{\partial u}{\partial z} = 0$ $\qquad \frac{\partial v}{\partial z} = 1$ $\qquad \frac{\partial w}{\partial z} = 3$

$$\begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$= 2x(3+2) + 2(3-1) + 0$

$= 2x(5) + 2(2)$

$= 10x + 4$ //

4. Find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ If $u = \frac{x^3y^3}{x^3+y^3}$

Sol. Given $u = \frac{x^3y^3}{x^3+y^3}$

$$\frac{\partial u}{\partial x} = \frac{(x^3+y^3)\frac{\partial}{\partial x}(x^3y^3) - x^3y^3\frac{\partial}{\partial x}(x^3+y^3)}{(x^3+y^3)^2}$$

$$= \frac{(x^3+y^3)(3x^2y^3) - x^3y^3(3x^2)}{(x^3+y^3)^2}$$

$$= \frac{\cancel{3x^5y^3} + 3x^2y^6 - \cancel{3x^5y^3}}{(x^3+y^3)^2}$$

$$= \frac{3x^2y^6}{(x^3+y^3)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^3+y^3)(3y^2x^3) - x^3y^3(3y^2)}{(x^3+y^3)^2}$$

$$= \frac{3x^6y^2 + \cancel{3y^5x^3} - \cancel{3y^5x^3}}{(x^3+y^3)^2}$$

$$= \frac{3x^6y^2}{(x^3+y^3)^2}$$

consider $x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$$x\cdot\frac{3x^2y^6}{(x^3+y^3)^2} + y\cdot\frac{3x^6y^2}{(x^3+y^3)^2}$$

=

$$= \frac{3x^3y^6 + 3x^6y^3}{(x^3+y^3)^2}$$

$$= \frac{3x^3y^3\cancel{(y^3+x^3)}}{(x^3+y^3)^{\cancel{2}}}$$

$$= 3\frac{x^3y^3}{x^3+y^3}$$

$$= 3u \; //$$

5. If $u = \log\frac{x^2+y^2}{x+y}$ S.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

$$\frac{\partial u}{\partial x} = \frac{1}{\frac{x^2+y^2}{\cancel{x+y}}} \frac{\left((x+y)\frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial x}(x+y)\right)}{(x+y)^{\cancel{2}}}$$

$$= \frac{(x+y)(2x) - (x^2+y^2)(1)}{(x+y)(x^2+y^2)}$$

$$= \frac{2x^2 + 2xy - (x^2+y^2)}{(x+y)(x^2+y^2)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\frac{x^2+y^2}{\cancel{x+y}}} \frac{\left((x+y)\frac{\partial}{\partial y}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial y}(x+y)\right)}{(x+y)^{\cancel{2}}}$$

$$= \frac{(x+y)(2y) - (x^2+y^2)(1)}{(x+y)(x^2+y^2)}$$

$$= \frac{2xy + 2y^2 - (x^2+y^2)}{(x+y)(x^2+y^2)}$$

Consider $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$$= x\left[\frac{2x^2+2xy-(x^2+y^2)}{(x+y)(x^2+y^2)}\right] + y\left[\frac{2xy+2y^2-(x^2+y^2)}{(x+y)(x^2+y^2)}\right]$$

$$= \frac{2x^3+2x^2y-x(x^2+y^2)}{(x+y)(x^2+y^2)} + \frac{2xy^2+2y^3-y(x^2+y^2)}{(x+y)(x^2+y^2)}$$

$$= \frac{2x^3+2x^2y-x^3-xy^2+2xy^2+2y^3-x^2y-y^3}{(x+y)(x^2+y^2)}$$

$$= \frac{x^3+2x^2y+2xy^2-xy^2-x^2y+y^3}{(x+y)(x^2+y^2)}$$

$$= \frac{x^2y+xy^2+x^3+y^3}{(x+y)(x^2+y^2)}$$

$$= \frac{x(x^2+y^2)+y(x^2+y^2)}{(x+y)(x^2+y^2)}$$

$$= \frac{\cancel{(x^2+y^2)}\,\cancel{(x+y)}}{\cancel{(x+y)}\,\cancel{(x^2+y^2)}}$$

$$= 1 \;//$$

$$\therefore\; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1 \;//$$

6 If $x = r\cos\theta \quad y = r\sin\theta \quad z = z$, find $\frac{\partial(r,\theta,z)}{\partial(x,y,z)}$

given that $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$

Sol. We are given $x = r\cos\theta \quad y = r\sin\theta \quad z = z$

and $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$

since $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} \cdot \frac{\partial(r,\theta,z)}{\partial(x,y,z)} = 1$

$\therefore$ we have $\frac{\partial(r,\theta,z)}{\partial(x,y,z)} = \frac{1}{r}$ //.

7 If $u = \log(x^2 + xy + y^2)$ P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$

Sol: Given $u = \log(x^2+xy+y^2) \rightarrow (1)$

Since u is not a homogeneous fun, we write (1) as

$e^u = (x^2+xy+y^2) = f(x,y)$

clearly $f(x,y)$ is a homogeneous funct of degree '2'

By Euler's theorem

$x\cdot\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 2\cdot e^u.$

$\Rightarrow x\cdot e^u\cdot\frac{\partial u}{\partial x} + y\cdot e^u\frac{\partial u}{\partial y} = 2e^u.$

$\therefore \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2}$

8 If $u = \sin^{-1}\left[\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right]$ S.T $\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}.$

Sol: Given function is not a homogeneous function.

Give function can be written as

$$\sin u = \left[\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right] = f(x,y)$$

Now it is a homogeneous function of degree 0

i.e $\boxed{n=0}$

By Euler theorem $x\cdot\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} = n\cdot f$.

$$x\cdot\frac{\partial}{\partial x}\sin u + y\cdot\frac{\partial}{\partial y}\sin u = 0\cdot f \Rightarrow x\cdot\cos u\cdot\frac{\partial u}{\partial x} + y\cos u\cdot\frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \cos u\left[x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y}\right] = 0 \quad \Rightarrow x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

$$\therefore x\frac{\partial u}{\partial x} = -y\cdot\frac{\partial u}{\partial y} \Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{-y}{x}\cdot\frac{\partial u}{\partial y}}$$

(9) If $z = xy^2 + x^2y$, $x = at^2$, $y = 2at$, find $\frac{dz}{dt}$

Sol: Give $z = xy^2 + x^2y$, $x = at^2$ & $y = 2at$.

$$\frac{\partial z}{\partial x} = y^2 + 2xy, \quad \frac{\partial z}{\partial y} = 2xy + x^2, \quad \frac{dx}{dt} = 2at \ \& \ \frac{dy}{dt} = 2a$$

$$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}.$$

$$= (y^2+2xy)(2at) + (2xy+x^2)(2a) = [4a^2t^2 + 2at^2\cdot 2at][2at] + [2at^2\cdot 2at + a^2t^4]2a$$

$$= \cancel{2ay^2t + 4xyat + 4axy + 2ax^2}$$

$$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^4$$

$$= 2a\cancel{[2at]^2t + 4[at^2][2at]at + 4a[at^2][2at]} + 2a[\cancel{at^2}]^2$$

$$= 16a^3t^3 + 10a^3t^4$$

$$= 2a^3t^3(8+5t)$$

$$= 2a\cancel{[4a^2t^2]t + 4(at^2)2a^2t^2 + (4a^2t^2)(2at) + 2a(a^2t^4)}$$

$$= \cancel{8a^3t^3 + 8a^3t^3 + 8a^3t^3 + 2a^3t^4 = 16a^3t^3 + 2a^3t[\ldots]}$$

(10) If $u = y^2 - 4ax$, $x = at^2$, $y = 2at$ find $\frac{du}{dt}$

Sol: $u = y^2 - 4ax$ $\quad x = at^2 \quad y = 2at$

$$\frac{\partial u}{\partial x} = -4a \ \Big| \ \frac{\partial u}{\partial y} = 2y \qquad \frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= -4a(2at) + 2y\cdot(2a)$$

$$= -8a^2t + 2(2at)(2a)$$

$$= -8a^2t + 8a^2t = 0$$

UNIT - V

ESSAY QUESTIONS

1) If $x = r\cos\theta$, $y = r\sin\theta$ then prove that

$$\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r} \text{ and } \frac{1}{r} \cdot \frac{\partial x}{\partial \theta} = r \cdot \frac{\partial \theta}{\partial x}.$$

sol: From the given equations we can write,

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$x^2+y^2 = r^2$$

$x = r\cos\theta \longrightarrow (1)$; $y = r\sin\theta \longrightarrow (2)$

$$\frac{(2)}{(1)} \Rightarrow \frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \frac{y}{x}$$

$$\tan\theta = \frac{y}{x}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x = r\cos\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \longrightarrow (3)$$

$$r^2 = x^2 + y^2$$

Differentiate w.r.t "x"

$$\cancel{2}r \frac{\partial r}{\partial x} = \cancel{2}x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{\cancel{r}\cos\theta}{\cancel{r}}$$

$$\frac{\partial r}{\partial x} = \cos\theta \longrightarrow (4)$$

From eqns ③ & ④ ,

$$\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$$

$$\frac{1}{r} \cdot \frac{\partial x}{\partial \theta} = r \cdot \frac{\partial \theta}{\partial x}$$

Consider, $\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta} [r\cos\theta]$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\Rightarrow \frac{1}{r} \cdot \frac{\partial x}{\partial \theta} = \frac{-\cancel{r}\sin\theta}{\cancel{r}}$$

$$\frac{1}{r} \cdot \frac{\partial x}{\partial \theta} = -\sin\theta = L.H.S$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$= \frac{1}{1+\cancel{\tan}\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{1}{\frac{x^2+y^2}{x^2}} \left(\frac{-y}{x^2}\right)$$

$$= \frac{-y}{x^2+y^2} = \frac{-\cancel{r}\sin\theta}{r\cancel{r}} = \frac{-\sin\theta}{r}$$

$$\Rightarrow r \cdot \frac{\partial \theta}{\partial x} = r \cdot \frac{-\sin\theta}{r}$$

$$r \cdot \frac{\partial \theta}{\partial x} = -\sin\theta \; ; \quad \therefore \frac{1}{r} \frac{\partial x}{\partial \theta} = r \cdot \frac{\partial \theta}{\partial x} \; //$$

2) If $x+y+z = u$, $y+z = uv$, $z = uvw$, then evaluate

(i) $\frac{\partial(x, y, z)}{\partial(u, v, w)}$ (ii) $J\left[\frac{u, v, w}{x, y, z}\right]$.

Sol :- Given $u = x+y+z \longrightarrow ①$

$uv = y+z \longrightarrow ②$

$uvw = z \longrightarrow ③$

From ②, $uv = y+z \Rightarrow y = uv - z = uv - uvw$ [using ③]

From ①, $u = x+y+z \Rightarrow x = u-(y+z)$

$\Rightarrow x = u - uv$ [using ②]

$\therefore \frac{\partial x}{\partial u} = \frac{\partial}{\partial u}[u-uv] = 1-v$, $\frac{\partial x}{\partial v} = -u$

$\frac{\partial x}{\partial w} = 0$

$\Rightarrow \frac{\partial y}{\partial u} = v - vw$, $\frac{\partial y}{\partial v} = u - uw$, $\frac{\partial y}{\partial w} = -uv$

$\Rightarrow \frac{\partial z}{\partial u} = vw$, $\frac{\partial z}{\partial v} = uw$ and $\frac{\partial z}{\partial w} = uv$

$$\therefore \frac{\partial(x, y, z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)\left[(u-uw)(uv)+u(uw)\right] + u\left[(v-vw)uv + uv(vw)\right]$$

$$= u^2 v$$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$$

(ii) Let $f_1 = u-x-y-z$, $f_2 = uv-y-z$ & $f_3 = uvw - z$

$$f\left(\frac{u,v,w}{x,y,z}\right)$$

We have, $J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)}$

$$= (-1)^3 \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} \div \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} \longrightarrow ①$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z \\ \partial f_3/\partial x & \partial f_3/\partial y & \partial f_3/\partial z \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} = -1 \longrightarrow ②$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} \partial f_1/\partial u & \partial f_2/\partial v & \partial f_1/\partial w \\ \partial f_2/\partial u & \partial f_2/\partial v & \partial f_2/\partial w \\ \partial f_3/\partial u & \partial f_3/\partial v & \partial f_3/\partial w \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}$$

$$= uv(u-0) = u^2 v \longrightarrow (3)$$

Substituting from (2) and (3) in (1) we get,

$$J\left(\frac{u, v, w}{x, y, z}\right) = (-1)\times(-1) \div u^2 v = \frac{1}{u^2 v}$$

$$\therefore J\left(\frac{u, v, w}{x, y, z}\right) = \frac{1}{u^2 v} \,//$$

3.Y If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta \text{ \& find } \frac{\partial(r, \theta, \phi)}{\partial(x, y, z)}.$$

Sol:- $x = r\sin\theta\cos\phi$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \;;\quad \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \;;\quad \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

$y = r\sin\theta\sin\phi$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi \;;\quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \;;\quad \frac{\partial y}{\partial \theta} = +r\sin\theta\cos\phi$$

$z = r\cos\theta$

$$\frac{\partial z}{\partial r} = \cos\theta \; ; \quad \frac{\partial z}{\partial \theta} = -r\sin\theta \; ; \quad \frac{\partial z}{\partial \phi} = 0$$

$$\begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$\sin\theta\cos\phi\,[-r\sin\theta(r\sin\theta\cos\phi)] - r\cos\theta\cos\phi\,[\cos\theta(r\sin\theta\cos\phi)]$$

$$-r\sin\theta\sin\phi\,[\sin\theta\sin\phi(-r\sin\theta) - \cos(r\cos\theta\sin\phi)]$$

$$= r\sin\theta\cos\phi\,(r\sin^2\theta\cos\phi) - r\cos\theta\cos\phi\,(-r\sin\theta\cos\theta\cos\phi) -$$

$$r\sin\theta\sin\phi\,(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)$$

$$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos\phi + r^2\sin^3\theta\sin\phi +$$

$$r^2\cos^2\theta\sin\theta\sin^2\phi$$

$$= r^2\sin^3\theta + r^2\cos^2\theta\sin\theta = r^2(\sin\theta)(\sin^2\theta + \cos^2\theta)$$

$$= r^2\sin\theta$$

$$\therefore \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$$

We are given $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$

Since, $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} \cdot \dfrac{\partial(r, \phi, \phi)}{\partial(x, y, z)} = 1$

$\therefore \dfrac{\partial(r, \theta, \phi)}{\partial(x, y, z)} = \dfrac{1}{r^2 \sin\theta}$.

4.4 (i) If $x = \frac{u^2}{v}$, $y = \frac{v^2}{u}$, find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol :- Given,

$x = \frac{u^2}{v}$; $y = \frac{v^2}{u}$

First, we find $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$

$x = \frac{u^2}{v}$

$\frac{\partial x}{\partial u} = \frac{2u}{v}$; $\frac{\partial x}{\partial v} = \frac{-u^2}{v^2}$

$y = \frac{v^2}{u}$

$\frac{\partial y}{\partial u} = \frac{-v^2}{u^2}$; $\frac{\partial y}{\partial v} = \frac{2v}{u}$

$$\begin{vmatrix} \frac{2u}{v} & \frac{-u^2}{v^2} \\ \frac{-v^2}{u^2} & \frac{2v}{u} \end{vmatrix}$$

$$\frac{2u}{v} \times \frac{2v}{u} - \frac{u^2}{v^2} \times \frac{v^2}{u^2}$$

$$4 - 1 = 3$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{3} \quad //$$

(ii) If $x = uv$, $y = u/v$ then find $\frac{\partial(x, y)}{\partial(u, v)}$

sol :- $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$x = uv$ $\qquad\qquad$ $y = \frac{u}{v}$

$\frac{\partial x}{\partial u} = v$ $\qquad\qquad$ $\frac{\partial y}{\partial u} = \frac{1}{v}$

$\frac{\partial x}{\partial v} = u$ $\qquad\qquad$ $\frac{\partial y}{\partial v} = \frac{-u}{v^2}$

$$\begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix}$$

$$v\left(\frac{-u}{v^2}\right) - \frac{u}{v} = \frac{-u}{v} - \frac{u}{v} = \frac{-2u}{v} \; //$$

(iii) If $x = uv$, $y = \frac{u}{v}$ verify that $\frac{\partial(x, y)}{\partial(u, v)} \times \frac{\partial(u, v)}{\partial(x, y)} = 1$

sol :- $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$x = uv$

$\frac{\partial x}{\partial u} = v$; $\frac{\partial x}{\partial v} = u$

$$y = \frac{u}{v}$$

$$\frac{\partial y}{\partial u} = \frac{1}{v} \quad ; \quad \frac{\partial y}{\partial v} = \frac{-u}{v^2}$$

$$\begin{vmatrix} v & u \\ 1/v & -u/v^2 \end{vmatrix} = \frac{-2u}{v} \longrightarrow (1)$$

$$x = uv \; ; \quad y = u/v$$

$$u = x/v$$

$$y = \frac{\frac{x}{v}}{v} \Rightarrow y = \frac{x}{v^2}$$

$$\Rightarrow v^2 = \frac{x}{y} \Rightarrow v = \sqrt{\frac{x}{y}}$$

$$u = \frac{x}{\frac{\sqrt{x}}{\sqrt{y}}} = \frac{\cancel{\sqrt{x}} \cdot \sqrt{x}}{\frac{\cancel{\sqrt{x}}}{\sqrt{y}}}$$

$$u = \sqrt{xy}$$

$$\Rightarrow u = \sqrt{xy} \; ; \quad v = \sqrt{\frac{x}{y}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$u = \sqrt{xy}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{xy}} \times y = \frac{y}{2\sqrt{xy}} \; ; \quad \frac{\partial u}{\partial y} = \frac{x}{2\sqrt{xy}}$$

$$v = \sqrt{x/y}$$

$$\frac{\partial v}{\partial x} = \frac{1}{2\sqrt{\frac{x}{y}}} \times \frac{1}{y} = \frac{1}{2y\sqrt{\frac{x}{y}}}$$

$$\frac{\partial v}{\partial y} = \frac{1}{2\sqrt{\frac{x}{y}}} \times \frac{-x}{y^2} = \frac{-x}{2y^2\sqrt{\frac{x}{y}}}$$

$$\Rightarrow \begin{vmatrix} \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} \\ \frac{1}{2y\sqrt{\frac{x}{y}}} & \frac{-x}{2y^2\sqrt{\frac{x}{y}}} \end{vmatrix}$$

$$\Rightarrow \frac{\cancel{y}}{2\sqrt{xy}} \times \frac{-x}{2\sqrt{\frac{x}{y}} \times y^{\cancel{2}}} - \frac{x}{2y\sqrt{\frac{x}{y}}\,2\sqrt{xy}}$$

$$\Rightarrow \frac{-x}{4\sqrt{xy}\sqrt{\frac{x}{y}}\cdot y} - \frac{x}{4y\sqrt{\frac{x}{y}}\sqrt{xy}}$$

$$\Rightarrow \frac{-\cancel{2}x}{{}_{2}\cancel{4}\sqrt{xy}\sqrt{\frac{x}{y}}\;y}$$

$$\Rightarrow \frac{-1\;v^{\cancel{2}}}{2u\cancel{v}} \quad \left[\because u = \sqrt{xy},\ v = \sqrt{\frac{x}{y}},\ v^2 = \frac{x}{y}\right]$$

$$\Rightarrow \frac{-v}{2u} \longrightarrow ②$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1 \quad [\text{from } ① \text{ \& } ②]$$

5) Find the maximum and minimum values of

$x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x = 0$

Sol:- Given,

$f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72$$

$$\frac{\partial f}{\partial y} = 6xy - 30y$$

$$\frac{\partial f}{\partial x} = 0 ; \quad 3x^2 + 3y^2 - 30x + 72 = 0 \longrightarrow ①$$

$$\frac{\partial f}{\partial y} = 0 ; \quad 6xy - 30y = 0$$

$$6y(x-5) = 0$$

$$6y = 0 \quad ; \quad x - 5 = 0$$

$$y = 0 \quad \quad x = 5$$

Sub $x = 5$ in eqn ①

$x = 5$; $3x^2 + 3y^2 - 30x + 72 = 0$

$$3(5)^2 + 3y^2 - 30(5) + 72 = 0$$

$$3(25) + 3y^2 - 150 + 72 = 0$$

$$75 + 3y^2 - 150 + 72 = 0$$

$$3y^2 - 3 = 0$$

$$3y^2 = 3$$

$$y = \pm 1 \quad (5, 1) \ \& \ (5, -1)$$

$y = 0$; $3x^2 + 3y^2 - 30x + 72 = 0$

$$3x^2 + 3(0)^2 - 30x + 72 = 0$$

$$3x^2 - 30x + 72 = 0$$

$$3(x^2 - 10x + 24) = 0$$

$$x^2 - 10x + 24 = 0$$

$$x^2 - 6x - 4x + 24 = 0$$

$$x(x-6) - 4(x-6) = 0$$

$$(x-4)(x-6) = 0$$

$$x = 4 ; x = 6$$

$$(4, 0) \quad (6, 0).$$

$\therefore$ The points are $(5, 1)\,(5, -1)\,(4, 0)\,(6, 0)$.

At point $(5, 1)$

$$l = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}(3x^2 + 3y^2 - 30x + 72)$$

$$= 6x - 30$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}[6xy - 30y]$$

$$= 6y$$

$$n = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(6xy - 30y)$$

$$= 6x - 30$$

At point (5, 1),

$$ln - m^2 = (6x - 30)(6x - 30) - (6y)^2$$

$$= (6(5) - 30)(6(5) - 30) - (6(1))^2$$

$$= (30 - 30)(30 - 30) - 6^2$$

$$= -36 < 0$$

At point (5, -1),

$$ln - m^2 = (6x - 30)(6x - 30) - (6y)^2$$

$$= (6(5) - 30)(6(5) - 30) - (6(-1))^2$$

$$= (30 - 30)(30 - 30) - 36 = -36 < 0$$

At point (4, 0),

$$ln - m^2 = (6x - 30)(6x - 30) - (6y)^2$$

$$= (6(4) - 30)(6(4) - 30) - (6(0))^2$$

$$= (24 - 30)(24 - 30)$$

$$= (-6)(-6) = 36 > 0$$

$$l = (6x - 30) = 6(4) - 30 = -6 < 0$$

It is minimum point

$$\text{Maximum value} = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

$$= 4^3 + 3(4)(0)^2 - 15(4)^2 - 15(0)^2 + 72(4)$$

$$= 64 - 240 + 288 = 112.$$

At point $(6, 0)$

$$ln - m^2 = (6x - 30)(6x - 30) - (6y)^2$$

$$= (6(6) - 30)(6(6) - 30) - (6(0))^2$$

$$= (36 - 30)(36 - 30)$$

$$= (6)(6) = 36 > 0$$

It is minimum point

$$l = (6x - 30) = 6(6) - 30$$

$$= 36 - 30 = 6 > 0$$

It is minimum point

Minimum value $= x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$$= (6)^3 + 3(6)(0) - 15(6)^2 - 15(0) + 72(6)$$

$$= 216 - 540 + 432$$

$$= 108 \; //$$

6. Find the rectangular parallelopiped of maximum volume that can be inscribed in a sphere.

Sol. Let 'a' (constant) be the radius of the given sphere. Also let x, y, z be the length, breadth and height of a rectangular parallelopiped inscribed in the given sphere.

The equation of the sphere is

$$x^2+y^2+z^2=a^2 \longrightarrow ①$$

Volume of the rectangular parallelopiped is

$$V = xyz \longrightarrow ②$$

From eq ①,

$$z=\sqrt{a^2-x^2-y^2} \longrightarrow ③$$

Substituting eq ③ in eq ②, we get

$$V = xy\sqrt{a^2-x^2-y^2}$$

$$\therefore V^2 = x^2y^2(a^2-x^2-y^2)$$

$$= x^2y^2a^2 - x^4y^2 - x^2y^4 \longrightarrow ④$$

Let $f(x,y) = V^2 = x^2y^2a^2 - x^4y^2 - x^2y^4$.

$$\therefore \frac{\partial f}{\partial x} = 2xy^2a^2 - 4x^3y^2 - 2xy^4$$

$$= 2xy^2(a^2-2x^2-y^2)$$

$$\frac{\partial f}{\partial y} = x^2(2y)a^2 - 2x^4y - 4x^2y^3$$

$$= 2x^2y(a^2-x^2-2y^2)$$

For V i.e; f to be maximum,

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 2xy^2(a^2 - 2x^2 - y^2) = 0$$

$$a^2 - 2x^2 - y^2 = 0 \quad (\because x \neq 0;\ y \neq 0) \longrightarrow (5)$$

And $\dfrac{\partial f}{\partial y} = 0 \Rightarrow a^2 - x^2 - 2y^2 = 0 \longrightarrow (6)$

eq (6) - eq (5) gives,

$$x^2 - y^2 = 0$$

$$x = y$$

From eq (5),

$$x = y = \frac{a}{\sqrt{3}}$$

From eq (3) we get,

$$z = \sqrt{a^2 - \frac{a^2}{3} - \frac{a^2}{3}} = \frac{a}{\sqrt{3}}$$

$$\therefore x = y = z = \frac{a}{\sqrt{3}}$$

$\therefore$ The critical point is $\left[\dfrac{a}{\sqrt{3}}, \dfrac{a}{\sqrt{3}}\right]$

Now,

$$r = \frac{\partial^2 f}{\partial x^2} = 2a^2y^2 - 12x^2y^2 - 2y^4;$$

$$s = \frac{\partial^2 f}{\partial x\,\partial y} = 4a^2xy - 8x^3y - 8xy^3;$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2a^2x^2 - 2x^4 - 12x^2y^2.$$

7: Find the maximum value of $u = x^2y^3z^4$ if $2x+3y+4z=a$

Sol. Given that,

function $F = x^2y^3z^4$.

$\phi = 2x+3y+4z = a$

$F(x,y,z) = f(x,y,z) + \lambda\phi(x+y+z)$

$= x^2y^3z^4 + \lambda(2x+3y+4z-a)$

$\frac{\partial F}{\partial x} = 2xy^3z^4 + 2\lambda = 0$

$\frac{\partial F}{\partial y} = 3x^2y^2z^4 + 3\lambda = 0$

$\frac{\partial F}{\partial z} = 4x^2y^3z^3 + 4\lambda = 0$

$\frac{\partial F}{\partial x} = 0$; $2xy^3z^4 + 2\lambda = 0$

$2xy^3z^4 = -2\lambda$

$-\lambda = xy^3z^4 \longrightarrow (1)$

$\frac{\partial F}{\partial y} = 0$; $3x^2y^2z^4 + 3\lambda = 0$

$3x^2y^2z^4 = -3\lambda$

$-\lambda = x^2y^2z^4 \longrightarrow (2)$

$\frac{\partial F}{\partial z} = 0$; $4x^2y^3z^3 + 4\lambda = 0$

$4x^2y^3z^3 = -4\lambda$

$-\lambda = x^2y^3z^3 \longrightarrow (3)$

At $\left[\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right]$,

$$r = 2a^2\left[\frac{a^2}{3}\right] - 12\cdot\frac{a^2}{3}\cdot\frac{a^2}{3} - 2\cdot\frac{a^4}{9}$$

$$= \frac{-8a^4}{9} < 0 \quad (\because a > 0)$$

$$s = 4a^2\left[\frac{a^2}{3}\right] - 8\cdot\frac{a^3}{3\sqrt{3}}\cdot\frac{a}{\sqrt{3}} - 8\cdot\frac{a}{\sqrt{3}}\cdot\frac{a^3}{3\sqrt{3}}$$

$$= \frac{4a^4}{3} - \frac{8a^4}{9} - \frac{8a^4}{9} = \frac{-4a^4}{9}$$

$$t = 2a^2\left[\frac{a^2}{3}\right] - 2\cdot\frac{a^4}{9} - 12\cdot\frac{a^2}{3}\cdot\frac{a^2}{3}$$

$$= \frac{-8a^4}{9}$$

Now,

$$rt - s^2 = \frac{64a^8}{81} - \frac{16a^8}{81}$$

$$= \frac{48a^8}{81} = \frac{16a^8}{27} > 0$$

Since $r < 0$ and $(rt - s^2) > 0$, therefore $f(x, y)$ i.e; V^2 and hence V is maximum at $\left[\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right]$. i.e; Volume is maximum for

$$x = y = z = \frac{a}{\sqrt{3}}.$$

$\therefore$ Inscribed rectangular parallelopiped in a cube.

Hence maximum volume of the rectangular parallelopiped =

$$xyz = \left[\frac{a}{\sqrt{3}}\right]^3 = \frac{a^3}{3\sqrt{3}} \text{ cu. units.}$$

8. Find the minimum value of $x^2+y^2+z^2$. given $x+y+z=3a$.

Sol. Given that,

function $f = x^2+y^2+z^2$ and $\phi = x+y+z-3a$

$$F = f(x,y,z) + \lambda\phi(x,y,z)$$

$$= (x^2+y^2+z^2) + \lambda(x+y+z-3a) \longrightarrow ①$$

Differentiate eq ① with respect to "x"

$$2x + \lambda(1) = 0$$

$$2x + \lambda = 0 \longrightarrow ②$$

Differentiate eq ① with respect to "y"

$$2y + \lambda(1) = 0$$

$$2y + \lambda = 0 \longrightarrow ③$$

Differentiate eq ① with respect to "z"

$$2z + \lambda(1) = 0$$

$$2z + \lambda = 0 \longrightarrow ④$$

From eq ②, eq ③, eq ④

$$x = -\frac{\lambda}{2} \;;\quad y = -\frac{\lambda}{2} \;;\quad z = -\frac{\lambda}{2}.$$

Substitute these values in $\phi(x,y,z) = x+y+z-3a$

$$= -\frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} - 3a$$

$$= -\frac{3\lambda}{2} - 3a$$

$$-\frac{3\lambda}{2} = 3a$$

Divide eq ① by eq ②

$$\frac{xy^3z^4}{x^2y^2z^4} = \frac{-\lambda}{-\lambda}$$

$$\frac{1}{x} \times y = 1$$

$$y = x$$

Divide eq ② by eq ③

$$\frac{x^2y^2z^4}{x^2y^3z^3} = \frac{-\lambda}{-\lambda}$$

$$\frac{1}{y} \times z = 1$$

$$z = y$$

$$\therefore \; x = y = z.$$

Substitute $x = y = z$ in $\phi(x, y, z)$

$$2x + 3x + 4x = a$$

$$9x = a$$

$$x = \frac{a}{9}$$

Similarly,

$$y = \frac{a}{9} \; ; \; z = \frac{a}{9}.$$

The maximum value $= x^2y^3z^4$

$$= \left[\frac{a}{9}\right]^2 \left[\frac{a}{9}\right]^3 \left[\frac{a}{9}\right]^4$$

$$= \left[\frac{a}{9}\right]^9$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(2x^3y - 2x^4y - 3x^3y^2)$$

$$= 6x^2y - 8x^3y - 9x^2y^2$$

$$= x^2y(6-8x-9y)$$

$$n = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$$

$$= 2x^3(1-x-3y)$$

$$\therefore ln - m^2 = 6xy^2(1-2x-y)\,2x^3(1-x-3y) - (x^2y)^2(6-8x-9y)^2$$

$$= (x^2y)^2\left[12(1-2x-y)(1-x-3y) - (6-8x-9y)^2\right]$$

For maximum and minimum;

$$\frac{\partial f}{\partial x} = 0 \Rightarrow x^2y^2(3-4x-3y) = 0$$

$$x = 0;\ y = 0 \text{ (or) } 3-4x-3y = 0$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow x^3y(2-2x-3y) = 0$$

$$x = 0;\ y = 0 \text{ (or) } 2-2x-3y = 0$$

The possible extreme of $f(x,y)$ are

$(x=0, y=0)$, $(x=0$ and $3-4x-3y=0)$, $(x=0$ and $2-2x-3y=0)$,

$(y=0$ and $2-2x-3y=0)$ and $(3-4x-3y=0$ and $2-2x-3y=0)$

i.e; $(0,0)$ $(0,1)$, $\left[0, \frac{2}{3}\right]$, $(1,0)$, $(0,1)$ and $\left[\frac{1}{2}, \frac{1}{3}\right]$

At all these points except $\left[\frac{1}{2}, \frac{1}{3}\right]$; $ln - m^2 = 0$ i.e, there is no extreme value.

$$\lambda = -2a.$$

Substitute 'λ' value in x, y, z

$$x = -\frac{\lambda}{2} = \frac{-(-2a)}{2} = a$$

Similarly,

$$y = a \; ; \; z = a$$

$\therefore$ The points are $(x, y, z) = (a, a, a)$

The minimum value $= x^2 + y^2 + z^2$

$$= a^2 + a^2 + a^2 = 3a^2.$$

9. Find the maximum and minimum values of the function $f(x, y) = x^3y^2(1-x-y)$.

Sol. We have,

$$f(x, y) = x^3y^2(1-x-y)$$

$$= x^3y^2 - x^4y^2 - x^3y^3$$

$$\therefore \frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$= x^2y^2(3 - 4x - 3y)$$

$$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$$

$$= x^3y(2 - 2x - 3y)$$

$$l = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$= 6xy^2(1 - 2x - y)$$

At $\left[\frac{1}{2}, \frac{1}{3}\right]$

$$ln - m^2 = \frac{1}{9 \cdot 64} > 0 \quad \&$$

$$l = 6\left[\frac{1}{2}\right]\left[\frac{1}{3}\right]^2\left[1 - 1 - \frac{1}{3}\right]$$

$$= -\frac{1}{9} < 0$$

$\therefore \left[\frac{1}{2}, \frac{1}{3}\right]$ is a pt of maximum.

$$\text{Maximum value} = f\left[\frac{1}{2}, \frac{1}{3}\right] = \left[\frac{1}{8} \cdot \frac{1}{9}\right]\left[1 - \frac{1}{2} - \frac{1}{3}\right]$$

$$= \frac{1}{72}\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{432}$$

10) S.T the functions $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$ $w = x + y + z$ are functionally related. Find the relation between them

Sol:

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$u = xy + yz + zx$ $\qquad v = x^2 + y^2 + z^2$ $\qquad w = x + y + z$

$$\frac{\partial u}{\partial x} = y + z \qquad \frac{\partial v}{\partial x} = 2x \qquad \frac{\partial w}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = x + z \qquad \frac{\partial v}{\partial y} = 2y \qquad \frac{\partial w}{\partial y} = 1$$

$$\frac{\partial u}{\partial z} = y + x \qquad \frac{\partial v}{\partial z} = 2z \qquad \frac{\partial w}{\partial z} = 1$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (y+z)(2y-2z) - (x+z)(2x-2z) + (y+x)(2x-2y)$$

$$= \cancel{2y^2} + \cancel{2zy} - \cancel{2zy} - \cancel{2z^2} - \cancel{2x^2} + \cancel{2zx} - \cancel{2xz} + \cancel{2z^2} + \cancel{2xy} - \cancel{2y^2} + \cancel{2x^2} - \cancel{2xy}$$

$$= 0$$

$\therefore$ x, y, z are functionally dependent.

$$w = x+y+z$$

Square on both sides

$$w^2 = (x+y+z)^2$$

$$= x^2+y^2+z^2+2xy+2yz+2zx$$

$$= x^2+y^2+z^2+2(xy+yz+zx)$$

$$\boxed{w^2 = v+2u.}$$